Journal of Statistical Physics, Vol. 87, Nos. 3/4, 1997

Inhomogeneous Kauffman Models at the Borderline Between Order and Chaos

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Received December 6, 1996; final December 16, 1996

We study a generalized Kauffman model where the interactions are no longer chosen according to a uniform probability distribution. It is shown that already slight deviations from the uniform distribution can drive the system into the chaotic phase, whereas the original model remains strictly in the ordered phase.

KEY WORDS: Disordered cellular automata; order–chaos; dynamical phase transitions.

1. INTRODUCTION

Kauffmann's model for genetic regulatory systems, introduced in the late sixties,[1,2] attracted much interest within the theoretical physics community[3-5] serving as a model of disordered cellular automata.

The dynamics of these sparsely interconnected systems is governed by a random mixture of all possible Boolean rules restricted to a few input variables. One remarkable result is that the dynamical behavior undergoes a sharp phase transition from *ordered* to *disordered* behavior when the system has more than two input variables per model unit.

One aim in this paper is to get some more insight into the principles that allow the networks to exhibit such profound order that they persist in the so-called frozen phase for connectivity $K \leqslant 2$.[3] Serving as order parameter is the "Lyapunov-like" normalized Hamming distance between two configurations $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ first introduced for Kauffman models by Derrida and Pomeau.[4]

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Table I. All Possible Automata Rules for $K=2$

Input	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
00	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
01	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
10	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
11	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

2. THE ORIGINAL KAUFFMAN MODEL

Kauffman considers a set of N interacting "binary genes" capable only of the two values 0 and 1 corresponding to the "off" and "on" states, respectively. Each unit is supposed to be influenced by K other units $j_1(i), j_2(i),..., j_K(i)$. These inputs are chosen at random among the N units of the network. Each site selects randomly a Boolean function f_i such that the dynamical time evolution is given by the parallel update rule

$$x_i(t+1) = f_i(x_{j_1(i)}(t),..., x_{j_K(i)}(t)) \tag{2.1}$$

Note that the Kaufmann model has been specified as follows:

- Each unit i receives *exactly* K inputs.
- The Boolean functions f_i are chosen at *random* according to a *uniform* distribution.

For $K=2$ all possible rules are listed in Table I. Prominent examples are the **XOR(6)**, **AND(8)** and **OR(14)** rule.

Since $K=2$ is a *critical* parameter value, one might expect that small fluctuations either in the connectivity or in the homogeneous choice of the Boolean rules can easily drive the system into the chaotic regime. In fact, it has already been shown[6] that an average connectivity of two inputs $\langle K \rangle = 2$ can lead to disordered behavior.

3. AN INHOMOGENEOUS KAUFFMAN NETWORK

Let us now drop Kauffman's assumption that the rules are democratically drawn with equal weight from a uniform distribution. We destroy the homogeneity with respect to the choice of the interactions and choose rule **1** with probability **p** and the 15 others with equal probability **(1 − p)/15**.

In order to study the stability of the network with respect to small changes in the initial conditions we define a commonly used order parameter, the normalized Hamming distance between two configurations $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$,

$$d(t)=\frac{1}{N}\sum_{i=1}^{N}|x_i^{(1)}(t)-x_i^{(2)}(t)| \tag{3.1}$$

This global variable $d(t)$ specifies the fraction of spins being different in two configurations. The crucial question is if an *initially* infinitesimally small distance $d(0)$ remains confined or eventually becomes finite in the large-time limit. Provided that T is big enough, $d(T)$ serves as a Lyapunov-like order parameter for our computer experiments.

Figure 1 depicts the time evolution of the distance $d(t)$ at concentration $p=0.4$ for four different initial distances chosen as $d(0)=0.01, 0.1, 0.2$, and 0.5, respectively. After a transient phase, the distance fluctuates around its mean value $\langle d(t)\rangle_{t=0}^{T}$. The fluctuations obey the central limit theorem, they are roughly Gaussian and decrease with increasing number of units N according to a $1/\sqrt{N}$ law. A surprising observation is that in marked contrast to the original Kauffman model, our *inhomogeneous* network has to be placed in the chaotic phase, since small initial distances obviously grow to a finite distance.

Figure 2 depicts the final distance $d(T)$ for $T=1000$ as a function of the concentration p for rule **1** with $d(0)=0.05$. For the concentration range $0<p<0.25$ the computer simulations cannot decide whether the system is in its chaotic or ordered phase. Presumably the system is close to the borderline separating ordered from chaotic behavior. However, with increasing

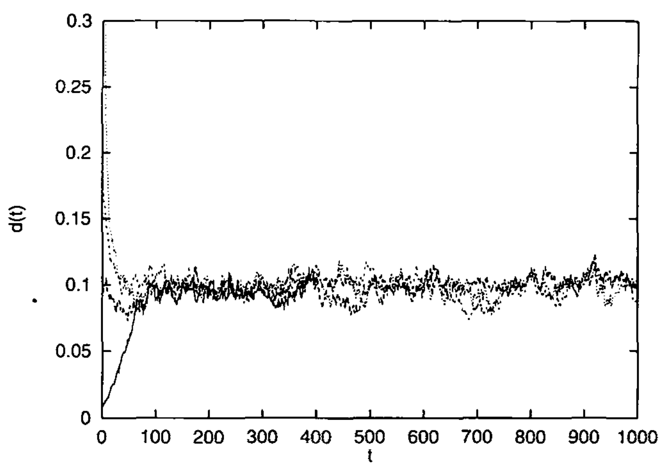

Fig. 1. Time evolution of the distance $d(t)$ for various initial distances $d(0)$.

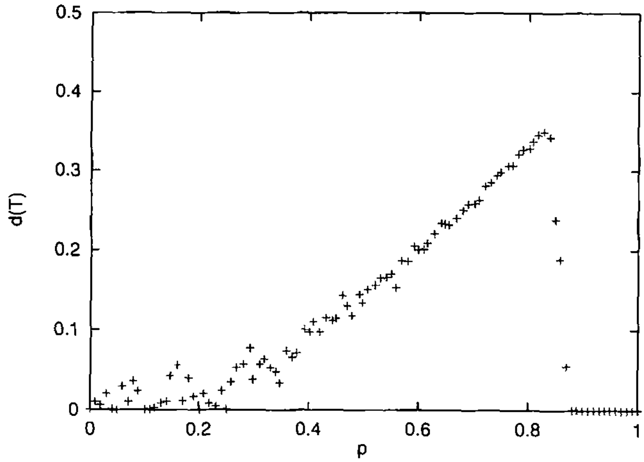

Fig. 2. Final distance $d(T=1000)$ as a function of the concentration p.

value of the concentration p we observe a continuous increase of $d(T)$, which reaches its maximum value $d_{max} \approx 0.354$ at $p_{max} \approx 0.82$. Then the distance decreases until distance zero is reached at $p_c \approx 0.88$. Figure 2 suggests that the transition is of second order, which has also been found for the homogeneous Kauffman model.[4] In fact, additional computer experiments confirm that for $10^6 < N < 10^7$ the linear decrease of $d(T)$ is almost independent of the number of units in the parameter range $p_{max} \approx 0.82$, $p_c \approx 0.88$.

Note that the degree of disorder is quite substantial, since the maximum possible degree is $d = 0.5$ corresponding to two configurations chosen completely at random.

4. BINARY MIXTURES OF AUTOMATA RULES

In order to get more insight into this unexpected behavior, we modify our $K=2$ model further and admit only *two* Boolean rules with weight $\mathbf{p}$ and $\mathbf{1-p}$, respectively. Such a simplified model system of binary mixtures of automata rules has already been studied by Hartmann and Vichniac[7] as well as by da Silva.[8] Their model lets the units reside on a two-dimensional square lattice with exclusively short-ranged interactions limited to the von Neumann and Moore neighborhood. Since they admit only the generalized **XOR** rule (parity) and the **AND** rule adapted to five and four inputs, respectively, the existence of a transition is obvious. Homogeneous automata, where each site obeys a K-input **XOR** rule, show maximum disorder $[d(\infty) = 0.5]$. On the other hand, homogeneous automata, where

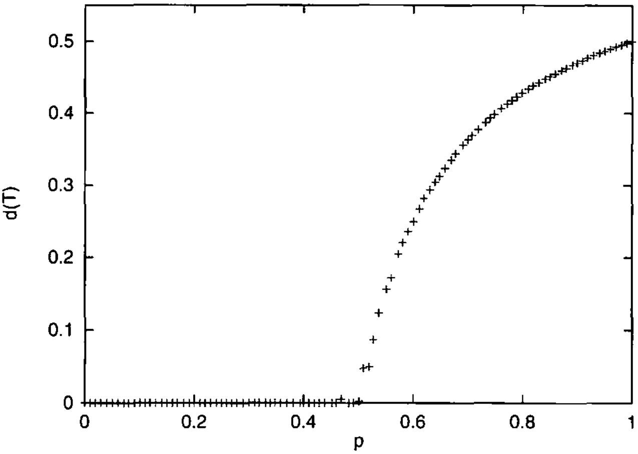

Fig. 3. Final distance $d(T = 1000)$ as a function of the concentration p.

Table II. Dynamical Behavior of the Binary Mixture[a]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	−	−	−	−	−	−	+	−	−	+	−	−	−	−	−	−
1	−	−	+	−	+	−	+	−	+	+	+	+	+	+	+	−
2	−	+	−	−	−	−	+	+	−	+	−	−	−	−	+	−
3	−	−	−	−	−	−	+	−	−	+	−	−	−	−	−	−
4	−	+	−	−	−	−	+	+	−	+	−	−	−	−	+	−
5	−	−	−	−	−	−	+	−	−	+	−	−	−	−	−	−
6	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
7	−	−	+	−	+	−	+	−	+	+	+	+	+	+	+	−
8	−	+	−	−	−	−	+	+	−	+	−	+	−	+	−	−
9	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
10	−	+	−	−	−	−	+	+	−	+	−	−	−	−	−	−
11	−	+	−	−	−	−	+	+	+	+	−	−	−	−	−	−
12	−	+	−	−	−	−	+	+	−	+	−	−	−	−	−	−
13	−	+	−	−	−	−	+	+	+	+	−	−	−	−	−	−
14	−	+	+	−	+	−	+	+	−	+	−	−	−	−	−	−
15	−	−	−	−	−	−	+	−	−	+	−	−	−	−	−	−

[a] −, Always regular; +, transition to chaos.

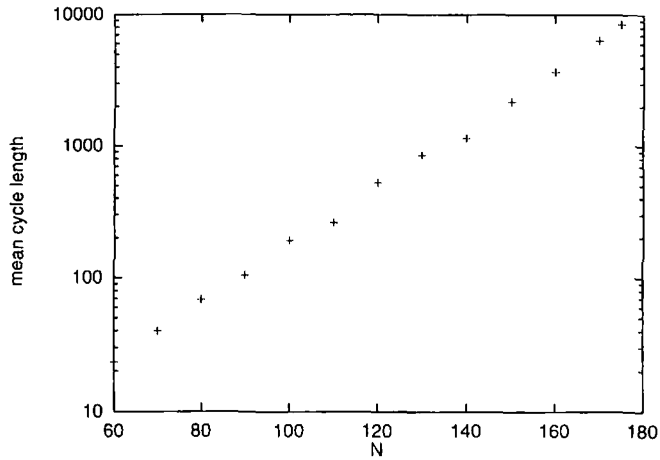

Fig. 4. Mean cycle length as a function of the number of units.

each site obeys a K-input **AND** rule, show extreme ordered behavior. After a short transient almost all sites are found to be frozen in the zero state.

In Fig. 3 we observe that in our model (which means with Kauffman's original long-ranged interactions) the final distance remains essentially zero for $p \leqslant 0.5$, where the **AND** rule dominates the **XOR** rule. The transition seems to take place at $p_c = 0.5$. With increasing concentration p the final distance increases rapidly to $d(T) = 0.5$, corresponding to the maximum degree of disorder, where only the **XOR** rule is present.

We now consider *all* possible pairs of $K = 2$ input automata and examine if a transition to chaos takes place or not (for any value of p). Table II demonstrates clearly that besides the expected transition for the **XOR** rule and its complement **rule 9** transitions are not uncommon at all in binary mixtures of $K = 2$ automata.

Figure 4 depicts a semilog plot of the mean cycle length of a system described by a binary mixture of **NOR(1)** and **AND(8)** automata at the concentration $p = 0.8$ as a function of the number of cells. As has usually been seen in the chaotic regime the increase of the cycle length is exponential in the number of units.

5. SUMMARY

We have shown that the profound order found in $K = 2$ Kauffman networks, where each unit receives exactly two inputs, must be largely attributed to the fact that the interactions are chosen according to a *uniform* distribution. We further demonstrated that this special choice constitutes a singular case, preventing the model from showing its full richness. This fact, however, might rise new biological discussions about the model.

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